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The performance of a stochastic model of individual buyer behavior relative to a set of gasoline diary panel data is explored. Use of the model for pricing decision making under a set of assumptions about competitive behavior in a market situation is discussed and illustrated through the use of numerical examples.

Application of a Modified Linear Learning Model of Buyer Behavior

INTRODUCTION

A number of widely differing stochastic models of individual brand choice have been suggested in the marketing literature. Massy, Montgomery, and Morrison [6] give a detailed review of this part of the literature. Most of these models have included only past purchase or time-trend type effects; controllable marketing variables have for the most part not been included. Thus, while these models are of considerable importance in providing insight into the mechanics of the individual purchase process, they are of limited use for decision making. A relatively smaller number of models (e.g., [3, 8]) have attempted to include controllable variables, thus potentially providing direct input into the marketing decision-making process. We will treat here the Modified Linear Learning Model (MLL), described in [4, 5].

The theoretical development of the model is considered in some detail in [5]. A brief review of the theoretical basis for the model will be presented, but the focus of the article will be on the performance of the model on a set of live data in the context of retail gasoline pricing, and on the normative or decision-making implications that can be derived from that example.

Although the examples used in this article are from the area of retail gasoline pricing, there is nothing explicit in the construction of the model which limits its use to this product. Further, it should not be assumed that this model will provide a unique theory of consumer behavior, whether in the gasoline purchasing area or elsewhere. Morrison [7] has argued that stochastic models of brand choice are unlikely to provide unique theories of brand choice; rather they can provide (perhaps very rough) approximations to buyer behavior which one may be sure can be shown to be statistically *incorrect*, given enough data. The strength of this model, then, lies in the fact that while it may provide insight into the consumer purchasing process, it more importantly may be used to aid in making pricing decisions.

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MODEL ASSUMPTIONS AND CHARACTERISTICS

A rather homogeneous product class, gasoline, will be used extensively for illustration but other examples include packaged soaps and detergents or cigarettes where it is intuitively acceptable (and frequently verified) that actual brand-to-brand differences are small, although perceived differences may be large. Assume the brands available in the market break down logically into two mutually exclusive and exhaustive subsets, "Premium" and "Standard." "Premium" will generally be associated with "majors" and "Standard" with "independents" in gasoline marketing. These groups are generally characterized by differences in price.

We state the assumptions we will be making explicitly as follows:

A1. The market breaks down naturally into two sets of brands, "Premium" and "Standard," in which the most important difference between groups is product price.

A2. The Premium brands can be characterized by a single price at a given time; the Standard brands can also be characterized by a single (generally lower than Premium) price at a given time.

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A3. If a particular customer is unaffected by price variation, then his purchasing behavior can be explained by the simple linear learning model (SLL), see [5].

A4. Changes in price are perceived by a customer immediately and price levels only directly affect customers during the particular period when they are in effect.

A5. Customers act rationally, and they all perceive Premium as "superior" to Standard. (The implication here is that if a customer exhibits no learning, i.e., if he is affected only by price, then he will almost always choose Premium over Standard when they are priced at the same level.)

A6. The effect of price variation can be explained totally by Premium-Standard price difference per gallon; actual Premium (or Standard) price level has no effect. This is a debatable assumption, but certainly evidence exists at least to support the hypothesis that price difference is the most important indicator of price effect [2]. This assumption is not critical and can be generalized to include a "fair price" concept [4].

A7. There exists across families some (prior) joint distribution of initial probability of Premium purchase, P_0, and of price sensitivity, C (roughly, the fraction of a purchaser's decision determined by price), which we will call $F_{C,P_0}(c, p)$. Families will be assumed to sample a particular C^* and P_0^* from this distribution. These distributions will be assumed to be independent and both in the Beta family.

A8. The parameters of the model, once chosen for a particular family, are constant over time. (Model is parameter-stationary).

A4 implies that a non purchase-habit-forming individual will be affected by price *only* in the period in which a price occurs. This does not preclude longer-term effects for "habit-forming" customers.

A7 details the method that has been selected for handling population heterogeneity. Thus, although it may be difficult or impossible to observe certain parameters for particular individuals, it is assumed that these parameters have some distribution across the population as a whole which one can infer as part of the parameter estimation procedure. This method of handling heterogeneity offers the sizable advantage of not requiring the measurement of specific causal factors.

A8 implies that each family samples from some (the same) $F_{C,P_0}(c,p)$ distribution to get a particular C^*, P_0^*. These are fixed for the family. The other parameters of the model are fixed across families and over time.

Following A1–A8 we will be dealing here with the observation and description of the purchase pattern of a single individual (customer, consumer, family—all will be used interchangeably here) over time. Let us define a stochastic process, $\{X_t\}$, $t = 0, 1, \ldots$ to describe this behavior. $X_t = 0$ or 1 at any discrete purchase time, t, and $X_t = 1$ implies a purchase of Premium, $X_t = 0$ implies a purchase of Standard. Note that t can be a purchase index also; i.e., $t = 4$ can refer to the fourth purchase by a customer after the start of the period.

Let us further define the following:

- δ_t = Premium price minus Standard price per gallon at t,
- $P_t = Pr\{X_t = 1\}$,
- C = price consciousness of customer—roughly the fraction of his behavior determined by price,
- $\phi(\delta)$ = value of the price response function when price difference $= \delta$. It will be near 1 for δ small (or negative) and will be near 0 for large δ.

Following the above definitions and A1–A8, our model (MLL) has the following form:

$$P_{t+1} = (1 - C)(\alpha + \beta X_t + \lambda P_t) + C\phi(\delta_{t+1}). \tag{2}$$

Note that δ_{t+1}, following A6, is assumed to be the sole determinant of a customer's price-buying behavior. And when $C = 0$ above, following A3, the model reduces to what we will call the simple linear learning model (SLL). When $C = 1$, the purchase is affected strictly by price and, following A5, $P_t \simeq 1$ if $\delta_t \simeq 0$ and $C = 1$; $1 \geq C \geq 0$ in general. We also assume for simplicity that ϕ is continuous and has derivatives of all orders.

PARAMETER ESTIMATION

The theory and the mechanical details of the parameter estimation procedure are included in the Appendix. To summarize briefly: we observe the actual frequencies that the various purchase strings (0110, for example) occur in the sample. Then we estimate, under the assumptions of the model, the theoretical frequency of those strings as functions of the parameters to be estimated. Values of those parameters are then calculated to minimize a chi-square statistic.

AN EXAMPLE FROM RETAIL GASOLINE MARKETING

Data Used

The data used in this study were collected from a gasoline diary panel in a midwestern United States market. The data covered a period from October 10, 1970 to July 18, 1971 and included 1074 total families, 654 of whom had 3 or more purchases of regular gasoline in each of the periods of study. Note that these 654 families were "heavy buyers," i.e., those with less than 3 purchases in any one period were eliminated. For families with more than three purchases, the first three purchases were retained. These 654 families were then divided randomly into 2 subpopulations with each family having a .5 chance of entering each population. The first subpopulation was used for estimation; the second was used for prediction. Table 1 gives the breakdown of the total population into these categories by purchase pattern in each

Table 1
OBSERVED DATA

	(1) Period	(2) Pattern	(3) Number in estimated sample (325)	(4) Number in test sample (329)	(5) Total (4) + (3)
	1	000	85	75	160
	1	001	38	46	84
	1	010	16	18	34
	1	011	40	55	95
$\delta = 2.47$	1	100	18	20	38
	1	101	22	24	46
	1	110	14	15	29
	1	111	92	76	168
	2	000	67	71	138
	2	001	41	38	79
	2	010	17	17	34
	2	011	43	40	83
$\delta = 2.24$	2	100	21	16	37
	2	101	23	24	47
	2	110	19	15	34
	2	111	94	108	202
	3	000	46	51	97
	3	001	16	18	34
	3	010	16	16	32
	3	011	26	34	60
$\delta = 1.74$	3	100	25	29	54
	3	101	19	25	44
	3	110	27	29	56
	3	111	150	127	277
	4	000	60	48	108
	4	001	26	22	48
	4	010	15	15	30
	4	011	39	41	80
$\delta = 1.99$	4	100	22	20	42
	4	101	28	23	51
	4	110	20	18	38
	4	111	115	130	255
	5	000	97	96	193
	5	001	43	56	99
	5	010	15	18	33
	5	011	46	45	91
$\delta = 2.79$	5	100	15	10	25
	5	101	18	23	41
	5	110	14	7	21
	5	111	77	74	151

of the five price-periods described later in the article. On each purchase occasion, the buyer recorded the brand purchased, price paid, grade of gasoline, and other information not used here. The first problem was to allocate brands of gasoline to a "Premium" category and a "Standard" category. Regular grade gasoline prices per gallon (as reported by the panel families) were calculated for each brand and a mean price per gallon over all brands was also calculated. The brands clustered into two groups—one above the average price and one below the average price. The borderline cases amounted to about 4% of the market volume and were allocated according to whether they were generally judged to have a "major" (Premium) or "independent" (Standard) national position. Our terminology may cause confusion in this gasoline marketing example but it has been selected to be consistent with that in [5].

Within a brand, the higher octane gasoline is often referred to as Premium, and the lower octane grade as Regular or Standard. This study was done solely with Regular or lower octane gasoline purchases—Premium and Standard will always refer to brand-groups here. Across the eight months, the Premium category averaged about 60% of the market volume.

Premium and Standard prices were required to allow calculation of the price response function, $\phi(\delta)$. Daily prices were calculated for Premium and Standard, and five three-week periods were selected, which had relatively stable price differences (δ). The periods along with associated average price differences and sample standard deviations are indicated below:

STUDY PERIODS

Period	*Starting date*	*Average price differences*	*Sample standard deviations*
1	Oct. 31, 1970	2.47¢	.31
2	Jan. 7, 1971	2.24¢	.39
3	Mar. 7, 1971	1.74¢	.22
4	Apr. 12, 1971	1.99¢	.17
5	June 6, 1971	2.79¢	.41

The periods were selected to have stable prices within periods while yielding as large a spread of price differences between periods as possible. Empirical response frequencies for three-purchase patterns starting in each of these time periods were then developed.

Parameter Estimates

Parameter estimates were made for MLL along with two other models, SLL and compound Bernoulli, for comparison. SLL is a special case of MLL with $C = 0$ for all families. The compound Bernoulli model assumes that each customer has a constant probability p of buying Premium and $1 - p$ of buying Standard in each period. p is assumed to be different for each customer, (see [6] for details). The compound Bernoulli is also a special case of MLL, with $C = 0$, $\alpha = \beta = 0$, $\lambda = 1$ for each consumer. Thus, these three models form a hierarchy and are a natural set for comparison.

The parameters were estimated numerically using a modified gradient search procedure. The parameter values are displayed in Table 2 along with MLL parameter estimates on the full sample (which equals estimating plus test sample). Table 3 compares the goodness of fit of these models, both for fitting the estimation data and for prediction. Along with the χ^2 values in that table, the p-values for each of the

Table 2
PARAMETER ESTIMATES

Parameter	*MLL (325)*	*SLL (325)*	*Bernouli (325)*	*MLL (654 families)*
$a_p(1)$	.836	.762	.640	.811
$b_p(1)$	.545	.600	.611	.522
$a_p(2)$	.761	.976	.969	.772
$b_p(2)$	.474	.671	.791	.451
$a_p(3)$	.773	.901	.955	.784
$b_p(3)$	.420	.365	.462	.440
$a_p(4)$	.808	.989	.967	.829
$b_p(4)$	.406	.552	.653	.429
$a_p(5)$	.858	.844	.542	.840
$b_p(5)$	.570	.795	.619	.540
a_c	1.71	—	—	1.90
b_c	3.81	—	—	4.11
$\phi(\delta_1)$	.278	—	—	.263
$\phi(\delta_2)$	.342	—	—	.355
$\phi(\delta_3)$	.679	—	—	.658
$\phi(\delta_4)$	.471	—	—	.487
$\phi(\delta_5)$	.189	—	—	.133
α	.018	.027	—	.021
β	.385	.177	—	.375
λ	.594	.699	—	.601

models are included, where $p = P_r\{\chi^2 \geq \text{observed } \chi^2 | \text{Model is true}\}$. For fitting MLL here, there are 15 degrees of freedom—40 response proportions with 35 independent components (each set of 8 parameters in each period must add to 1) minus 20 degrees of freedom for parameter estimation [1]. The degrees of freedom for the other models are calculated similarly. For prediction, no parameters are estimated and 35 degrees of freedom are available for each of the models. Comparing p-values indicates that MLL describes or fits the data best. And comparing the χ^2 and p-values for prediction indicates MLL also predicts much better than the other models.

If our goal were to make a case for MLL as the model of consumer gasoline purchasing behavior, then, clearly, many detailed comparisons between models would be in order. However, our goal is more modest: we wish to show that MLL adequately describes some test data, adequately predicts test results, and does so better than similar models of the same class. These points should be clear now, and it is thus beyond our needs to go into finer statistical detail about significance levels and powers of tests between alternative model hypotheses.

Table 3
COMPARISON OF MODEL FITS

Model	*Degrees of freedom*	χ^2	*p-value*
MLL-fitting	15	14.4	.50
Prediction	35	25.1	.80
SLL-fitting	22	61.9	<.01
Prediction	35	69.5	<.01
Bernoulli-fitting	25	66.3	<.01
Prediction	35	119.5	<.01
MLL-fitting-full sample	15	7.6	.90

Interpretation of Model Parameters

An interpretation will be provided here of the parameters estimated on the basis of full sample of 654 families. Recall that each period can have a different prior distribution of P_0 (which we assumed to be in the Beta family; see the Appendix). Thus the values of $a_p(1)$, $b_p(1)$ imply that the prior distribution of P_0 in period 1 is $B(a_p(1), b_p(1))$. Similarly, a_c, b_c refer to the parameters of the prior (Beta) distribution of C, assumed constant across periods. Recalling that $\delta_1 = 2.47$, it is clear that $\phi(\delta_1) = \phi(2.47)$, referring to the price-function estimate which applies to the first period.

We can make some concrete statements about the implications of this model. First observe the $\{a_p, b_p\}$; all a_p, $b_p < 1$, which implies that the prior distribution of P_0 is bimodal, peaking at $P_0 = 0$ and $P_0 = 1$. This may suggest trying either a segmentation approach or a more general, bimodal distribution in further work. Thus, in each case, a randomly selected customer is quite likely to have either a very high or very low initial probability of purchasing Premium. The means of the 5 prior distributions are all reasonably near the 60% or so market share observed in this market for the Premium brands; see Table 4 for details.

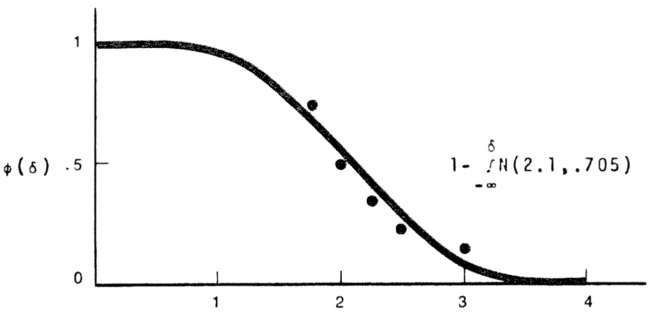
The prior distribution for C, price consciousness, on the other hand, has a_c, $b_c > 1$ and is therefore unimodal. Thus the "average" family is likely to have price consciousness of about 32%, and 90% of all families will have price consciousness between 5% and 60%. These results imply that in this market the average family's purchase decision is affected about 32% by the price difference in the market and the rest by "learning" or brand purchase-habit formation.

The values of $\phi(\delta)$ are plotted against δ in Figure 1. (Recall here that $\phi(\delta)$ can be thought of as the probability of buying Premium for a consumer who is totally price conscious, i.e., for whom $C = 1$.) Also sketched in that figure is a curve of the form $1 - \int_{\infty}^{\delta} N(\mu, \sigma)$, i.e., 1-cumulative normal distribution curve. The fit (for $\mu = 2.1$, $\sigma = .705$) appears acceptable, but, with only five points, many other curves (with perhaps different normative implications)

Table 4
MLL STATISTICS

Distribution	*Mean*	*Variance*
Prior $P_0(1)$	.608	.102
Prior $P_0(2)$	.631	.104
Prior $P_0(3)$	.640	.103
Prior $P_0(4)$	.659	.099
Prior $P_0(5)$	.609	.100
Prior C_0	.316	.031

Figure 1
PRICING RESPONSE FUNCTION



could fit these data as well. The calculation of these parameters was made numerically, minimizing an objective function of the sum of squared differences:

$$\text{find } \mu, \sigma \text{ to}$$

$$\min \sum_{i=1}^{5} \left[\phi(\delta_i) - 1 + \int_{-\infty}^{\delta_i} N(\mu, \sigma) \right]^2.$$

The fit using a normal approximation was slightly better than the fit from a log-normal approximation in terms of the sum of squared residuals. The implications of the normal approximation with the parameters previously mentioned are of some interest: a purely price-conscious purchaser will be indifferent between Premium and Standard at a difference of 2.1¢/gallon, will have a 90% chance of buying Standard at a difference of 3¢/gallon, and a 10% chance of buying Standard at a difference of 1.2¢/gallon.

Pricing Decision Making: An Aside

We are considering behavior in a two-brand market where we wish to set the price of one of the brands. The competitor can be "cooperative," "competitive," "indifferent." We can look at different objectives as well as different planning horizons. We may wish to consider various constraints—perhaps maximizing along the direction of one objective while satisficing along another. And the market may be price inelastic or elastic in one of many different ways.

Thus, there is no "right problem" for analysis but rather a set of problems, some of which might be more closely related to a particular real-life situation than others. As an example of a particular problem, assume that Standard is nonprice competitive, i.e., Standard sets its price and allows Premium complete freedom in setting δ, the price difference. Define q = unit price of Standard product (assumed constant over time), d = cost per unit of product for Premium, $V(q, \delta)$ = total market demand for the product as a function of q, δ.

Premium must set its price. Suppose Premium's objective is to maximize long-term expected profit and the only strategy permissible is a single product price. It can then be seen that Premium's problem becomes:

$$\text{find } \delta \text{ to}$$

$$\max\,(q + \delta - d)\, V(q, \delta) \cdot \int_0^1 \frac{(1 - c)\alpha + c\phi(\delta)}{1 - (1 - c)(\beta + \lambda)} dF_c(c) \qquad (3)$$

where $dF_C(c)$ is the distribution of price consciousness across families and the expression in the integral is $\lim_{t\to\infty} E(P_t)$. Lilien [4] shows how to develop solution strategies for this and more complex cases.

Some Pricing Policy Examples

In this section we use the model to develop some numerical examples of optimal pricing policies for Premium. We will be using the values of the parameters developed in the "Parameter Estimates" section for the full sample of 654 families.

Example 1: This example will consider pricing against an indifferent competitor in a homogeneous, price inelastic market, where $C = c$ is fixed. We assume that Standard sets a particular price (q) and sees unit profit $q - d$, where d = product cost to both Premium and Standard. Premium's problem can be seen from (3) to be:

$$\text{find } \delta \text{ to}$$

$$\max Z = (q + \delta - d)\frac{(1 - c)\alpha + c\phi(\delta)}{1 - (1 - c)(\beta + \lambda)}.$$

Figure 2 plots Z against δ for various values of C, where Z is long-term expected profit per gallon as in (3). The important thing to notice is that for a wide range of values of C, the optimum differences lie in the .9–1.4¢/gallon range.

Example 2: Let us consider the case where, through some process, Premium and Standard have agreed to cooperate and to maximize their joint long-term

Figure 2
LONG-TERM UNIT PROFIT VS. PRICE DIFFERENCE

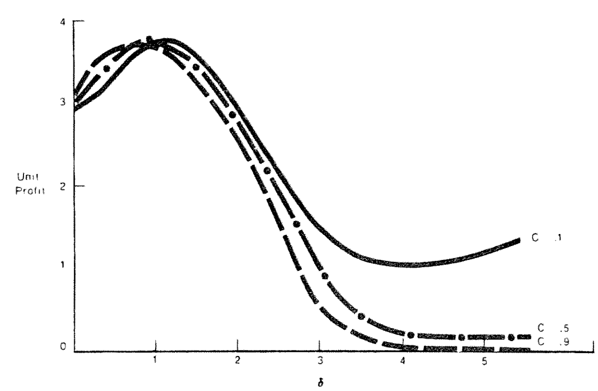
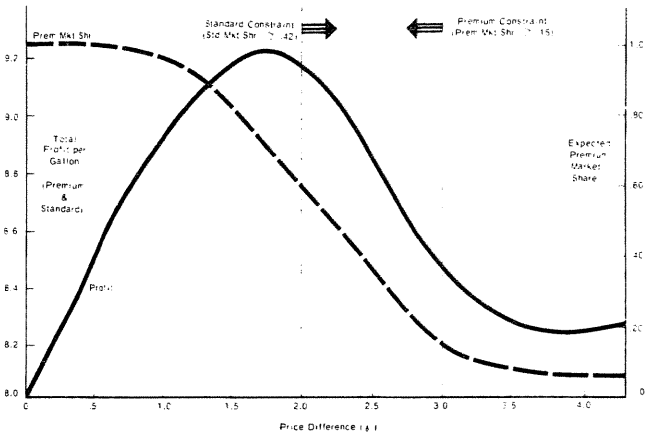


Figure 2 Long Term Unit Profit vs. Price Difference

Figure 3

TOTAL MARKET LONG-TERM EXPECTED PROFIT/GALLON



expected profit in this (still) price-inelastic market. Assume also there is some upper bound $(q - d)^*$ on Standard's profit margin, set implicitly. (This might be the highest price that can be set without government intervention). Then for N families with associated c_l, $l = 1, \ldots N$, the problem can be formulated as:

find δ to

$$\max Z = N(q - d) + \delta \sum_{l=1}^{N} \frac{(1 - c_l)\alpha + c_l \phi(\delta)}{1 - (1 - c_l)(\beta + \lambda)}.$$

If, in addition, Premium and Standard are only willing to cooperate as long as they maintain a minimum expected long-term market share, we must add the following constraint:

$$K_2 \geq \lim_{t \to \infty} E[P_t | P_{01} = p_{01}, P_{02} = p_{02}, \ldots P_{0N} = p_{0N}, \delta] \geq K_1$$

For a particular example we have set Standard profit margin = 8¢/gallon and sampled from a Beta (1.9, 4.1) population to get 10 c_j's = .28, .21, .18, .37, .41, .55, .23, .27, .11, .52.

Long-term expected profit/gallon sold in the market as a function of price difference is plotted in Figure 3. Also plotted is a particular constraint example. Note that without constraints, the maximum long-term profit per gallon is 9.3¢, achieved at δ = 1.7¢ and long-term Premium market share = 72.5%. However, if we assume that Standard will not accept a market share less than .42, or Premium accept a market share less than .15, the feasible price difference region turns out to be [2, 3]. Here the optimal difference = 2¢ and associated expected total profit = 9.1¢/gallon.

CONCLUSIONS AND USES

On the basis of the empirical work discussed in this article, it appears that a model such as MLL might well be of benefit in the area of marketing decision making. The model was shown to fit the data well and to lead to frequently nontrivial but understandable pricing strategies in the cases explored.

To use the model most effectively we feel that a pricing decision maker in a particular market should then: (1) gather purchase and price data and estimate model parameters, (2) test the fit of the model to see if it is a sufficiently accurate representation of the market situation, (3) design an experiment to verify parameter estimates and also to infer mode of competitive reactions to price changes, (4) run sensitivity studies on critical variables, and then use these data to develop the type of information that can be of aid in making pricing decisions.

Some possible future developments extending this study might include: (1) model modifications to include more than two brands, (2) model generalizations to include more controllable market variables or nonconstant family parameters, (3) extensions of the model to other product classes, (4) tests in other market areas and/or at different times, (5) extension of these normative ideas to other stochastic models of buyer behavior. Few stochastic models of buyer behavior include decision variables as part of the structure. It has been demonstrated here that it is possible and useful to develop stochastic models which involve decision variables.

APPENDIX
PARAMETER ESTIMATION METHODS

Let us assume we observe purchase sequences of length 2 (with $C = c$, $P_0 = p$), and define: $\gamma_{ij}(c, p)$ = conditional probability of purchase sequence i, j with $C = c$; $P_0 = p$; $i, j = 0, 1$. Then assuming the family is randomly sampled from the population, the unconditional probability of observing sequence i, j is as follows:

$$\text{(4)} \qquad \Pi_{i,j} = Pr\{\text{sequence } i, j\} = \int_0^1 \int_0^1 \gamma_{ij}(c, p)\, dF_{c,P_0}(c, p)$$

As an example, when $i = j = 1$:

$$\text{(5)} \qquad \Pi_{1,1} = (\alpha + \beta)\, EP_0 + (\phi(\delta) - \alpha - \beta) E(CP_0) + \lambda EP_0^2 - \lambda E(CP_0^2).$$

There are eight parameters in (5) (α, β, λ, $\phi(\delta)$, and four expectations—EP_0, etc.) and only three degrees of freedom for estimation with two-purchase sequences. Thus we cannot estimate the parameters without either more purchases or more structure.

Suppose we isolate k different periods of time long enough so that most customers make at least (say) three purchases and during which time price differences are reasonably constant. Call these prices $\delta_1 \ldots \delta_K$, the index referring to the constant-price period.

We must now estimate $\phi(\delta_1), \ldots, \phi(\delta_K)$ along with the other parameters. To cut down the size of the problem, we assume some structure for $F_{C,P_0}(c, p)$. Assume each family conceptually samples P_0 and C independently from some prior distribution which we will assume here is from the Beta family; i.e.

$$dF_C(c) = B(a_c, b_c)$$

$$= \frac{\Gamma(a_c + b_c)}{\Gamma(a_c)\Gamma(b_c)} c^{a_c - 1}(1 - c)^{b_c - 1}, \; c \epsilon [0, 1].$$

Similarly $dF_{P_0}(p) = B(a_p, b_p)$. Then α, β, λ and the prior distribution of C (two parameters) remain the same in each of the K periods. However, variation in the period-to-period prior distribution of P_0 (two parameters per period or $2K$ parameter) must be included to eliminate interperiod correlation and to account for marketing disturbances between periods. This is necessitated by the fact that the *same* families are observed in each of the k periods. We must then estimate $3K + 5$ parameters: α, β, λ, a_c, b_c (for prior $F_C(c)$) plus $(\phi(\delta_k), a_p(k), b_p(k))$, $k = 1, \ldots K$.

If we consider strings of length 3, there are eight distinct response sequences, yielding seven degrees of freedom per period or $7K$ degrees of freedom for estimation. We lose one degree of freedom for each parameter estimated, so K response periods yield $4K - 5$ degrees of freedom for estimation (e.g., 15 degrees of freedom for $K = 5$). We use this information as follows:

Assuming response strings of length 3, define:

$Z_{l,k}$ = observed proportion of individuals in the population exhibiting response sequence ($l = 1$ implies 000, $l = 2$ implies 001, etc.) during period k; $l = 1, \ldots 8$, $k = 1 \ldots K$,

$\Pi_l(\delta_k)$ = expected proportion of population exhibiting response sequence l during period k. This is a function of the parameters to be estimated, and

N = number of individuals in the study.

A measure of discrepency between these observed and expected frequencies is given by the following statistic:

$$\chi^2 = N \sum_{l=1}^{8} \sum_{k=1}^{K} \frac{[Z_{lk} - \Pi_l(\delta_k)]^2}{\Pi_l(\delta_k)}. \tag{6}$$

We then calculate values of the parameter to minimize (6). Assuming the $Z_{l,k}$'s are at least approximately independent [5], and the model true, it can be shown [1] that as $N \to \infty$, (6) is asymptotically distributed as chi-square with $7K - m$ degrees of freedom where m is the number of parameters, $7K - m = 4K - 5$ in this case. This information is then used to test the "goodness of fit" of the model.

Minimizing (6) with respect to the parameter vector is not possible analytically and numerical, nonlinear optimization methods have to be relied on. And as (6) is unlikely to be concave in the parameters, the parameter space has to be searched quite carefully to be certain of a "global" optimum.

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